International Journal of Theoretical Physics, Vol. 46, No. 1, January 2007 (© 2006)
DOI: 10.1007/s10773-006-9189-x

From Schwarzschild to Kerr due to Black Hole Quantum Tunnelling

Li Gao,[1] Hongbao Zhang,[1] and Wenbiao Liu[1,2]

Received March 1, 2006; accepted March 20, 2006
Published Online: November 28, 2006

Hawking radiation can be viewed as a process of quantum tunnelling near black hole horizon. When a particle with angular momentum tunnels across the event horizon of Schwarzschild black hole, the black hole will change into a Kerr black hole. The emission rate of the massless particles with angular momentum is calculated, and the result is consistent with an underlying unitary theory.

KEY WORDS: Hawking radiation; black hole; Bekenstein–Hawking entropy; quantum unitary theory; Painleve coordinates.

1. INTRODUCTION

Stephen Hawking's significant discovery (Hawking, 1975; Gibbons and Hawking, 1977) that black hole radiates thermally set up a disturbing and difficult problem: what happens to information during black hole evaporation? This is also called as the information loss paradox. Hawking's result also implies the loss of unitarity, or even the breakdown of quantum mechanics (Hawking, 1976). Treating Hawking radiation as quantum tunnelling across the event horizon, Parikh and Wilkzek proposed a semiclassical method to calculate the emission rate, and give these problems a physical explanation (Parikh and Wilczek, 2000; Parikh, 2004a,b). Their key point is to find a coordinate system well-behaved at the event horizon, and the barrier is created just by the outgoing particle itself.

They have calculated the tunnelling of massless particles from Schwarzschild black hole and Reissner–Norstrom black hole, and their result is consistent with an underlying unitary theory. Following this method, the radiation from AdS black hole and de Sitter cosmological horizon were also studied (Hemming and Keski-Vakkuri, 2001; Medved, 2002; Parikh, 2002). All these spherically symmetric

[1] Department of Physics, Institute of Theoretical Physics, Beijing Normal University, Beijing 100875, China.
[2] To whom correspondence should be addressed at Department of Physics, Institute of Theoretical Physics, Beijing Normal University, Beijing 100875, China; e-mail: wbliu@bnu.edu.cn.

investigations are successful. Because of the complexity of the stationary axisymmetric Kerr black hole (Kerr, 1963), the tunnelling effect must be investigated in the dragging coordinate system. Jingyi Zhang and Zheng Zhao have extended this method from spherical state case to the general axisymmetric Kerr and Kerr–Newman black holes, and even have studied more general massive and charged tunnelling (Zhang and Zhao, 2005a,b,c,d). The results are successful. For axisymmetric stationary black holes, the picture is: a particle do tunnel out of black hole, the barrier is created by the outgoing particle itself. If the total energy and angular momentum must be reserved, the outgoing particle must tunnel out a radial barrier to an observer resting in dragging coordinate system. Actually, they discussed a special case during which $a = \frac{J}{M}$ of the hole keeps invariant. In addition there were many other related works (Wu and Jiang, in press a,b; Jiang and Wu, 2006a,b; Arzano, Medved and Vagenas, 2005; Medved and Vagenas, 2005a,b; Angheben, Nadalini, Vanzo, and Zerbini, 2005; Setare and Vagenas, 2005, 2004; Vagenas, 2003, 2002a,b, 2001) in the last several years that have attracted the interest of the scientific community. Among these papers there are very new works (this shows that the field is still active), there are also older and important works that the new ones have been based on, and finally there are examples of the Parikh-Wilczek methodology utilized in cosmological frameworks.

In this paper, we will discuss the particle with angular momentum tunnels across the event horizon of Schwarzschild black hole, and then Schwarzschild black hole becomes a Kerr black hole. The result we got is also consistent with an underlying unitary theory.

2. PAINLEVE COORDINATES

To describe tunnelling, we need a coordinate system that is regular at the horizon, which is called as Painleve coordinates (Painleve, 1921). The line element of Kerr black hole is as follows (Wald, 1984)

$$ds^2 = -\left(1 - \frac{2Mr}{\rho^2}\right) dt_s^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \left[(r^2 + a^2) sin^2\theta + \frac{2Mra^2 \sin^4\theta}{\rho^2}\right] d\varphi^2 - \frac{4Mra\sin^2\theta}{\rho^2} dt_s d\varphi, \quad (1)$$

where $\rho^2 = r^2 + a^2\cos^2\theta$, $\Delta = r^2 + a^2 - 2Mr$. When a particle with energy ω, angular momentum J tunnels out of Schwarzschild black hole's event horizon, the black hole will carry angular momentum $(-J)$, and become to a Kerr black hole, with $a = \frac{-J}{M-\omega}$, $\rho^2 = r^2 + (\frac{-J}{M-\omega})^2 \cos^2\theta$, $\Delta = r^2 + (\frac{-J}{M-\omega})^2 - 2(M-\omega)r$.

The event horizon of this black hole should be given by

$$r_H = M - \omega + \sqrt{(M-\omega)^2 - \left(\frac{-J}{M-\omega}\right)^2}. \tag{2}$$

From Eq. (1), we can see that there is coordinate singularity at the event horizon. To obtain the Painleve-like coordinates, we first investigate the dragging coordinate system. Let

$$\frac{d\varphi}{dt_s} = -\frac{g_{03}}{g_{33}} = \Omega, \tag{3}$$

so the line element of Kerr black hole becomes

$$\begin{aligned} ds^2 &= \hat{g}_{00} dt_s^2 + g_{11} dr^2 + g_{22} d\theta^2 \\ &= -\frac{\rho^2 \Delta}{\left[r^2 + \left(\frac{-J}{M-\omega}\right)^2\right]^2 - \Delta\left(\frac{-J}{M-\omega}\right)^2 \sin^2\theta} dt_s^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2. \end{aligned} \tag{4}$$

In fact, the line element Eq. (4) represents a 3-dimensional hypersurface in 4-dimensional Kerr spacetime. Although the dragging coordinate system does not have singularity at the event horizon, this coordinate system is not yet what we want, because it is not flat Euclidean space in radial to constant-time slices. So we make the following transformation

$$dt_s = dt + F(r,\theta)dr + G(r,\theta)d\theta, \tag{5}$$

where $F(r,\theta)$ and $G(r,\theta)$ are two functions of r and θ to be determined. As a corollary, demanding constant-time slices to be flat, we can obtain the condition

$$g_{11} + \widehat{g}_{00}\, F(r,\theta)^2 = 1. \tag{6}$$

So we can write down the Painleve-like line element as

$$\begin{aligned} ds^2 &= \widehat{g}_{00} dt^2 + 2\sqrt{\widehat{g}_{00}(1-g_{11})}\, dt dr + dr^2 + [\widehat{g}_{00} G^2(r,\theta) \\ &\quad + g_{22}] d\theta^2 + 2\widehat{g}_{00} G(r,\theta)\, dt d\theta + 2\sqrt{\widehat{g}_{00}(1-g_{11})} G(r,\theta) dr d\theta, \end{aligned} \tag{7}$$

where

$$\widehat{g}_{00} = -\frac{\rho^2\Delta}{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta}, \qquad g_{11} = \frac{\rho^2}{\Delta}, \qquad g_{22} = \rho^2. \tag{8}$$

For massless particles which move along radial null geodesics, we can consider particle tunnelling as an ellipsoid shell. To conserve the symmetry of Kerr spacetime, the particle still should be an ellipsoid shell during the tunnelling process, that is to say, the particle does not have motion in θ direction. Under these conditions

$(d\theta = ds^2 = 0)$, we have

$$\dot{r} = \frac{\pm\rho^2 - \sqrt{\rho^2(\rho^2 - \Delta)}}{\sqrt{(r^2 + a^2)^2 - \Delta a^2 \sin^2\theta}}, \tag{9}$$

where $\dot{r} = \frac{dr}{dt}$, the $(+)$ sign corresponds to an outgoing geodesic and $(-)$ sign corresponds to an ingoing geodesic.

3. TUNNELLING ACROSS THE HORIZON

To ensure the conservation of energy and angular momentum, we fix the total mass and angular momentum of spacetime, and allow the hole mass and angular momentum to fluctuate. Then, when a particle with energy ω', angular momentum J' tunnels out, the black hole's mass and angular momentum will become $M - \omega'$ and $(-J')$ respectively. The radial coordinate of the event horizon will be

$$r'_+ = M - \omega' + \sqrt{(M - \omega')^2 - \left(\frac{-J'}{M - \omega'}\right)^2}, \tag{10}$$

and the dragging angular velocity of Kerr black hole will be

$$\Omega'_H = \frac{a}{(r'_+)^2 + a^2} = \frac{\frac{-J'}{M-\omega'}}{2(M - \omega')^2 + 2(M - \omega')\sqrt{(M - \omega')^2 - \left(\frac{-J'}{M-\omega'}\right)^2}}. \tag{11}$$

In the semiclassical limit, we can apply the WKB formula, the emission rate is

$$\Gamma \sim \exp(-2\mathrm{Im}S). \tag{12}$$

In the dragging coordinate system, the coordinate φ is an ignorable coordinate in the Lagrangian function $\mathcal{L}$. To eliminate this freedom completely, the imaginary part of action should be written as

$$\begin{aligned} \mathrm{Im}S &= \mathrm{Im}\int_{t_i}^{t_f} (\mathcal{L} - p_\varphi \dot{\varphi})dt = \mathrm{Im}\left[\int_{r_i}^{r_f} p_r dr - \int_{\varphi_i}^{\varphi_f} p_\varphi d\varphi\right] \\ &= \mathrm{Im}\left[\int_{r_i}^{r_f}\int_0^{p_r} dp'_r dr - \int_{\varphi_i}^{\varphi_f}\int_0^{p_\varphi} dp'_\varphi d\varphi\right], \end{aligned} \tag{13}$$

where p_r and p_φ are canonical momentums conjugate to r and φ, respectively.

To proceed the calculation, we apply Hamilton's equation

$$\dot{r} = \left.\frac{dH}{dp_r}\right|_r = \frac{d(M - \omega')}{dp_r} = \frac{-d\omega'}{dp_r}, \qquad \dot{\varphi} = \left.\frac{dH}{dp_\varphi}\right|_\varphi = \frac{\Omega'_H dJ'}{dp_\varphi}. \tag{14}$$

Thinking of

$$d\varphi = \Omega dt_s = \Omega(dt + F(r,\theta)dr) = \frac{\Omega}{\dot{r}}(1 + \dot{r}F(r,\theta))dr, \tag{15}$$

and

$$\dot{\varphi} = \frac{d\varphi}{dt} = \Omega(1 + \dot{r}F(r,\theta)), \tag{16}$$

putting Eq. (14) into Eq. (13), we have

$$\begin{aligned}\mathrm{Im}S &= \mathrm{Im}\left[\int_{r_i}^{r_f}\int_{M}^{M-\omega}\frac{d(M-\omega')}{\dot{r}}dr - \int_{r_i}^{r_f}\int_{0}^{J}\frac{\Omega'_H}{\dot{r}}dJ'dr\right]\\ &= \mathrm{Im}\left\{\int_{r_i}^{r_f}\left[\int_{(0,0)}^{(\omega,J)}\frac{d\omega'}{\dot{r}} - \frac{\Omega'_H}{\dot{r}}dJ'\right]dr\right\}.\end{aligned} \tag{17}$$

When the particle tunnels out of the event horizon, $\dot{r}$ will become to

$$\dot{r} = \frac{\pm\rho'^2 - \sqrt{\rho'^2(\rho'^2 - \Delta')}}{\sqrt{(r^2 + a'^2)^2 - \Delta' a'^2 \sin^2\theta}}, \tag{18}$$

where $a' = \frac{-J'}{M-\omega'}$, $\rho'^2 = r^2 + (\frac{-J'}{M-\omega'})^2\cos^2\theta$, $\Delta' = r^2 + (\frac{-J'}{M-\omega'})^2 - 2(M-\omega')r$. Putting Eq. (18) into Eq. (17), we have

$$\begin{aligned}\mathrm{Im}S = -\mathrm{Im}\Bigg\{\int_{r_i}^{r_f}\Bigg[\int_{(0,0)}^{(\omega,J)} &-\frac{\sqrt{(r^2 + a'^2)^2 - \Delta' a'^2\sin^2\theta}}{\rho'^2(r - r'_+)(r - r'_-)}\\ &\times[\rho'^2 + \sqrt{\rho'^2(\rho'^2 - \Delta')}]d\omega' + \frac{\sqrt{(r^2 + a'^2)^2 - \Delta' a'^2\sin^2\theta}}{\rho'^2(r - r'_+)(r - r'_-)}\\ &\times[\rho'^2 + \sqrt{\rho'^2(\rho'^2 - \Delta')}]\Omega'_H dJ'\, dr\Bigg]\Bigg\}.\end{aligned} \tag{19}$$

The initial location r_i roughly corresponds to the site of pair creation, which should be slightly inside the Schwarzschild black hole's horizon $r_i \approx 2M$, and the final radius r_f is slightly outside of the Kerr black hole's horizon $r_f \approx M - \omega + \sqrt{(M-\omega)^2 - (\frac{-J}{M-\omega})^2}$. Because the horizon shrinks, r_f is actually less than r_i. Back-reaction results in a shift of the horizon radius, and the finite separation between the initial and final radius is the classically-forbidden region, which is the barrier. We can see that $r = r'_+$ is a pole, the integral can be done by deforming the contour around the pole, then switching the order of the integration in Eq. (19), we get

$$\begin{aligned}
\mathrm{Im}S &= \pi\left[\int_{(0,0)}^{(\omega,J)} -\frac{2\left[(r'_+)^2+\left(\frac{-J'}{M-\omega'}\right)^2\right]}{r'_+-r'_-}d\omega' + \frac{2\left[(r'_+)^2+\left(\frac{-J'}{M-\omega'}\right)^2\right]}{r'_+-r'_-}\Omega'_H dJ'\right]\\
&= \pi\int_0^{\omega} -\frac{2\left[2(M-\omega')^2+2(M-\omega')\sqrt{(M-\omega')^2-\left(\frac{-J'}{M-\omega'}\right)^2}\right]}{2\sqrt{(M-\omega')^2-\left(\frac{-J'}{M-\omega'}\right)^2}}d\omega'\\
&\quad+\int_0^{J}\frac{2(M-\omega')^2+2(M-\omega')\sqrt{(M-\omega')^2-\left(\frac{-J'}{M-\omega'}\right)^2}}{\sqrt{(M-\omega')^2-\left(\frac{-J'}{M-\omega'}\right)^2}}\\
&\quad\times\frac{\frac{-J'}{M-\omega'}}{2(M-\omega')^2+2(M-\omega')\sqrt{(M-\omega')^2-\left(\frac{-J'}{M-\omega'}\right)^2}}dJ'\\
&= \pi\left[\int_{(0,0)}^{(\omega,J)} -\frac{2(M-\omega')^2+2(M-\omega')\sqrt{(M-\omega')^2-\left(\frac{-J'}{M-\omega'}\right)^2}}{\sqrt{(M-\omega')^2-\left(\frac{-J'}{M-\omega'}\right)^2}}d\omega'\right.\\
&\quad\left.+\frac{\frac{-J'}{M-\omega'}}{\sqrt{(M-\omega')^2-\left(\frac{-J'}{M-\omega'}\right)^2}}dJ'\right]\\
&= -\pi\left[(M-\omega)^2+(M-\omega)\sqrt{(M-\omega)^2-\left(\frac{-J}{M-\omega}\right)^2}-2M^2\right], \qquad (20)
\end{aligned}$$

where we have considered Eq. (10) and Eq. (11). The tunnelling rate is therefore

$$\begin{aligned}
\Gamma &\sim \exp(-2\mathrm{Im}S)\\
&= \exp\left\{2\pi\left[(M-\omega)^2+(M-\omega)\sqrt{(M-\omega)^2-\left(\frac{-J}{M-\omega}\right)^2}-2M^2\right]\right\}.
\end{aligned} \qquad (21)$$

The difference of the Bekenstein–Hawking entropy before and after emission is

$$\Delta S_{BH} = S_{BH2} - S_{BH1}$$

$$
\begin{aligned}
&= \pi\left[(r_f)^2 + \left(\frac{-J}{M-\omega}\right)^2\right] - \pi(2M)^2 \\
&= 2\pi\left[(M-\omega)^2 + (M-\omega)\sqrt{(M-\omega)^2 - \left(\frac{-J}{M-\omega}\right)^2} - 2M^2\right],
\end{aligned}
\tag{22}
$$

so we obtain

$$
\Gamma \sim \exp(-2\mathrm{Im}S) = \exp(\Delta S_{BH}), \tag{23}
$$

which is also consistent with an underlying unitary theory.

4. CONCLUSIONS AND DISCUSSIONS

We have calculated the emission rate of massless particles with angular momentum tunnelling across the event horizon of Schwarzschild black hole. After radiation, it becomes a Kerr black hole. According to “No hair theory,” the result we got can also offer a possible mechanism to deal with the information loss paradox because the spectrum is not pure thermal yet. In previous papers (Zhang and Zhao, 2005a,b), axisymmetric black hole have been studied only when a keeps constant. Changing from Schwarzschild to Kerr should be the simplest case when a varies. After this, we will have the capability to study more general radiation including changed a in Kerr or Kerr–Newman spacetimes in the future. The result we get here provides further evidence to support Parikh’s quantum tunnelling method.

ACKNOWLEDGMENTS

This research is supported by the National Natural Science Foundation of China (Grant No. 10373003, 10475013) and the National Basic Research Program of China (Grant No. 2003CB716302).

REFERENCES

Angheben, M., Nadalini, M., Vanzo, L., and Zerbini, S. (2005). *JHEP* **0505**, 014; hep-th/0503081.

Arzano, M., Medved, A. J. M., and Vagenas, E. C. (2005). *JHEP* **0509**, 037; hep-th/0505266.

Gibbons, G. W., and Hawking, S. W. (1977). *Physical Review D* **15**, 2752.

Hawking, S. W. (1975). *Communications in Mathematical Physics* **43**, 199.

Hawking, S. W. (1976). *Physical Review D* **14**, 2460.

Hemming, S., and Keski-Vakkuri, E. (2001). *Physical Review D* **64**, 044006.

Jiang, Q. Q., Wu, S. Q., and Cai, X. (2006a). *Physical Review D* **73**, 064003; hep-th/0512351.

Jiang, Q. Q., and Wu, S. Q. (2006b). *Physics Letter B* **635**, (2006) 151; hep-th/0511123].

Kerr, R. P. (1963). *Physical Review Letters* **11**, 237.

Medved, A. J. M. (2002). *Physical Review D* **66**, 124009.

Medved, A. J. M. (2002). *Class Quant Grav* **19**, 589; hep-th/0110289.

Medved, A. J. M., and Vagenas, E. C. (2005a). *Modern Physics Letters A* **20**, 1723; gr-qc/0505015.

Medved, A. J. M., and Vagenas, E. C. (2005b). *Modern Physics Letters A* **20**, 2449; gr-qc/0504113.

Painleve, P. (1921). *Comptes Rendus de l'Academie des Sciences, Serie I (Mathematique)* **173**, 677.

Parikh, M. K. (2002). *Physics Letters B* **546**, 189; hep-th/0204107.

Parikh, M. K. (2004a). Energy conservation and Hawking radiation. hep-th/0402166.

Parikh, M. K. (2004b). *International Journal of Modern Physics* **D13**, 2351; hep-th/0405160.

Parikh, M. K., and Wilczek, F. (2000). *Physical Review Letters* **85**, 5042; hep-th/9907001.

Setare, M. R., and Vagenas, E. C. (2005). *International Journal of Modern Physics A* **20**, 7219; hep-th/0405186.

Setare, M. R., and Vagenas, E. C. (2004). *Physics Letters B* **584**, 127; hep-th/0309092.

Vagenas, E. C. (2001). *Physics Letters B* **503**, 399; hep-th/0012134.

Vagenas, E. C. (2002a). *Physics Letters B* **533**, 302; hep-th/0109108.

Vagenas, E. C. (2002b). *Modern Physics Letters A* **17**, 609; hep-th/0108147.

Vagenas, E. C. (2003). *Physics Letters B* **559**, 65; hep-th/0209185.

Wald, R. M. (1984). *General Relativity*, Univercity of Chicago Press, Chicago.

Wu, Q., and Jiang Q. Q. (in press a). Hawking radiation of charged particles as tunneling from higher dimensional Reissner-Nordstrom-de Sitter black holes. hep-th/0603082.

Wu, Q., and Jiang Q. Q. (in press b). Remarks on Hawking radiation as tunneling from the BTZ black holes. hep-th/0602033.

Zhang, J., and Zhao, Z. (2005a). *Modern Physics Letters A* **20**, 1673.

Zhang, J., and Zhao, Z. (2005b). *Physics Letters B* **618**, 14.

Zhang, J., and Zhao, Z. (2005c). *Nuclear Physics B* **725**, 173.

Zhang, J., and Zhao, Z. (2005d). *JHEP* **10**, 055.